

CHIRAL PERTURBATION THEORY PREDICTIONS FOR $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ A. BRAMON^{a)}, P. GOSDZINSKY^{a,b,c)} and S. TORTOSA^{a)}

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ABSTRACT

The $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay is discussed in the general context of Chiral Perturbation Theory (ChPT), assuming that the low-energy constants (counter-terms) are saturated by vector-meson resonances. The $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ amplitude can be separated in two distinct pieces: the inner bremsstrahlung, $A^{(IB)}$, and the structure dependent (or direct emission), $A^{(SD)}$, amplitudes. The former – which essentially contains the same physics as $A(\eta \rightarrow \pi^+\pi^-\pi^0)$ – is found to dominate over the second one – which looks more interesting from the ChPT point of view.

Introduction

At low energies, Chiral Perturbation Theory (ChPT) provides an accurate description of the strong and electroweak interactions of pseudoscalar mesons, P , in terms of a perturbative series expansion [1]. At lowest order, ChPT is essentially equivalent to Current Algebra (CA) [2] but, at higher orders, loop effects appear restoring unitarity and giving rise to both finite corrections and divergences. The former are known to improve the lowest order (or CA) predictions, while the latter require the introduction of counterterms and destroy the renormalizability of the theory. The finite part of these counterterms, in turn, can be related to other successful aspects of hadron physics, such as the classical notion of Vector Meson Dominance (VMD). In this note we study the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay in the context of ChPT with VM dominated counterterms.

The amplitude for the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay contains two distinct pieces receiving two distinct types of contributions, $A(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) = A^{(IB)} + A^{(SD)}$. The first piece – the inner bremsstrahlung amplitude $A^{(IB)}$ – proceeds through the hadronic and G-parity violating $\eta \rightarrow \pi^+\pi^-\pi^0$ decay, accompanied by the emission of an isovector (positive G-parity) photon. This part of the amplitude presents an infrared divergence and it is obviously dominated by soft photon emission. It can be deduced from Low's theorem and, therefore, $A^{(IB)}$ essentially contains the same physics as the purely hadronic $\eta \rightarrow \pi^+\pi^-\pi^0$ amplitude carefully discussed by Gasser and Leutwyler [1]. The second piece – the structure dependent (or direct emission) amplitude $A^{(SD)}$ – looks *a priori* more interesting from the ChPT point of view. It involves an isoscalar (negative G-parity) photon and proceeds without the suppression due to G-parity violation. Therefore there is no obvious reason to assume that this G-parity conserving part of the amplitude, $A^{(SD)}$, has to be smaller than the first, G-parity violating one, $A^{(IB)}$, except for the low-energy end of the photonic spectrum.

The same conclusion was reached long ago by Singer [3] in his VMD calculation, as well as in the more detailed CA analysis of ref. [4], where $A^{(IB)}$ was predicted to be larger than $A^{(SD)}$ only for photon energies $E_\gamma \leq 15$ MeV, well below its maximum value $E_\gamma^{max} \simeq 120$ MeV. These two and other old theoretical calculations for $A^{(SD)}$ (based on CA and/or VMD) [3, 4, 5] lead to

$$\frac{\Gamma^{(SD)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)}{\Gamma(\eta \rightarrow \pi^0\gamma\gamma)} = 0.23 \%, 0.24 \%, 0.28 \%, \quad (1)$$

although another similar (but oversimplified) analysis [6] lead to a negligible $A^{(SD)}$ and to smaller values for the ratio above. Combining the predictions (1) with present day data for the eta meson [7] one obtains

$$\Gamma^{(SD)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) \simeq 2 \times 10^{-3} \text{ eV}, \quad B.R.^{(SD)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) \simeq 1.6 \times 10^{-6}. \quad (2)$$

Up to now, this structure dependent part of the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay has not been detected experimentally and only an upper limit – at 90 % of confidence level – is known [8].

$$\Gamma^{(SD)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) \leq 6 \times 10^{-4} \Gamma(\eta \rightarrow all) \simeq 0.72 \text{ eV}. \quad (3)$$

According to this discussion, the Saturne η -factory and the Daphne ϕ -factory could in principle produce enough etas to allow for detection and analysis of the structure dependent part of the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ amplitude. This prompted us to perform the corresponding ChPT calculation following the lines of ref. [9], where the analog (and similarly complicated) $\eta \rightarrow \pi^0\gamma\gamma$ decay was considered.

We will describe the nonet of pseudoscalar mesons P in terms of the $SU(3)$ octet and singlet matrices

$$P_8 = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2}{\sqrt{6}}\eta_8 \end{pmatrix}, \quad P_1 = \frac{1}{\sqrt{3}}\eta_1 \mathbf{I}, \quad (4)$$

which appear in the ChPT lagrangian through the parametrization

$$\Sigma \equiv \Sigma_8 \Sigma_1 = \Sigma_1 \Sigma_8 = \exp \left(\frac{2i}{f} (P_8 + P_1) \right), \quad (5)$$

with $f = 132 \text{ MeV}$ [1, 7]. Following the lines of ref. [9] and the reviews [10], we will adopt a rather simplified treatment of the singlet component of the η in ChPT. More precisely, we will assume that the physical η particle participates from nonet symmetry with an $\eta - \eta'$ mixing given by

$$\eta = \cos \theta \eta_8 - \sin \theta \eta_1 \simeq \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} - s\bar{s}), \quad \sin \theta \simeq -1/3, \quad (6)$$

as deduced from conventional $\eta - \eta'$ phenomenology and from more recent treatments in the ChPT context [10, 12, 13].

The lowest order lagrangian of ChPT (order two in particle four-momenta or masses, $O(p^2)$) is

$$L_2 = \frac{f^2}{8} \text{tr} (D_\mu \Sigma D^\mu \Sigma^\dagger + \chi \Sigma^\dagger + \chi^\dagger \Sigma), \quad (7)$$

apart from an extra singlet mass term for η_1 that should be added to account for the $U(1)_A$ problem, as discussed in [10]. The first term in (7) contains the covariant derivative $D_\mu \Sigma \equiv \partial_\mu \Sigma + ie A_\mu [Q, \Sigma]$, with the photon field A_μ and the quark charge matrix Q

$[Q = \text{diag}(2/3, -1/3, -1/3)]$. The non-derivative terms in Eq. (7), with $\chi = \chi^\dagger = B \mathcal{M}$, contain the quark mass matrix \mathcal{M} [$\mathcal{M} = \text{diag}(m_u, m_d, m_s)$] and lead to

$$\frac{1}{2}B = \frac{m_K^2}{m_u + m_s} = \frac{m_\pi^2}{m_u + m_d} = \frac{\Delta m_K^2}{m_d - m_u}, \quad (8)$$

where

$$\Delta m_K^2 \equiv (m_{K^0}^2 - m_{K^+}^2)_{QCD} = (6.2 \pm 0.5) \times 10^{-3} \text{ GeV}^2 \quad (9)$$

is an estimate for the QCD (or non-photonic) contribution to the squared $K^0 - K^+$ mass difference. The latter plays a central role in both $A(\eta \rightarrow \pi^+ \pi^- \pi^0)$ and the inner bremsstrahlung part of $A(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma)$. Its numerical value in (9) is an average between the result [1] $\Delta m_K^2 = (m_{K^0}^2 - m_{K^+}^2 - m_{\pi^0}^2 + m_{\pi^+}^2) = 5.3 \times 10^{-3} \text{ GeV}^2$, following from Dashen's theorem, and independent estimates [14] (including improved versions of Dashen's theorem [15]) leading to Δm_K^2 in the range $(6.5 - 7.0) \times 10^{-3} \text{ GeV}^2$.

The next order lagrangian, $O(p^4)$, contains the Wess-Zumino term (the anomalous sector) [16] and a series of ten (non-anomalous) counterterms identified and studied by Gasser and Leutwyler [1],

$$L_4 = L_{WZ} + \sum L_i L_4^{(i)}. \quad (10)$$

The only pieces of L_{WZ} relevant for our purposes are the ones containing the anomalous $PPP\gamma$ and $PPPP$ couplings, i.e.,

$$-\frac{e}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} A_\mu \text{tr}(Q \partial_\nu \Sigma \partial_\alpha \Sigma^\dagger \partial_\beta \Sigma \Sigma^\dagger - Q \partial_\nu \Sigma^\dagger \partial_\alpha \Sigma \partial_\beta \Sigma^\dagger \Sigma) \quad (11)$$

and

$$-\frac{2}{15\pi^2 f^5} \epsilon^{\mu\nu\alpha\beta} \text{tr}(P \partial_\mu P \partial_\nu P \partial_\alpha P \partial_\beta P), \quad (12)$$

respectively. The finite parts of the ten low-energy constants $L_i, i = 1, \dots, 10$ are real and have been fixed by experimental data [1]. Alternatively, they can be approximatively deduced assuming that they are saturated by the exchange of known meson resonances [10, 12, 13, 17], thus justifying the phenomenological success of conventional VMD. Fixing the renormalization mass-scale around these resonance masses ($\mu = M_\rho$, for instance), the finite, renormalized values for L_i are small enough to justify the convergence of the perturbative series. This last remark obviously does not apply to L_{WZ} , generating anomalous processes with theoretically well-defined coupling strengths.

The inner bremstrahlung amplitude $A^{(IB)}$

As previously stated, this part of the amplitude for $\eta(P) \rightarrow \pi^+(p_+) \pi^-(p_-) \pi^0(p_0) \gamma(q)$ can be related to the purely hadronic one, $A(\eta \rightarrow \pi^+ \pi^- \pi^0)$, by using Low's theorem. Either through this theorem or by explicit calculation, one obtains the following, conveniently

factorized expression

$$A^{(IB)} = -\frac{B(m_d - m_u)}{3\sqrt{3}} \frac{e}{f^2} \left(\frac{\epsilon p_+}{qp_+} - \frac{\epsilon p_-}{qp_-} \right) \left(1 + 2 \frac{m_\eta^2 - 3Pp_0}{m_\eta^2 - m_\pi^2} + U + V + W \right), \quad (13)$$

where the (infrared divergent) factor $e \left(\frac{\epsilon p_+}{qp_+} - \frac{\epsilon p_-}{qp_-} \right)$, containing the photon momentum q and polarization ϵ , comes from photon radiation by external pions, and the remaining factors correspond essentially to $A(\eta \rightarrow \pi^+ \pi^- \pi^0)$ [1].

The lowest order, $O(p^2)$, contribution proceeds through the tree level diagrams shown in Fig. 1a, b and c, with vertices of the L_2 lagrangian (7). At this lowest order, all η - η' mixing effects are ignored and the amplitude turns out to be given by eq.(13) with $U = V = W = 0$. The two diagrams in Fig. 1a give the dominant contribution corresponding to the first term in the last parenthesis. The second term comes from the three diagrams in Fig. 1b with a four-pseudoscalar vertex generated exclusively by the derivative part in L_2 . The remaining part of these three diagrams, generated by the massive couplings in L_2 , cancels with the contributions from Fig. 1c. However, this lowest order – but otherwise exact – expression is expected to underestimate the $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$ decay rate since the corresponding lowest order amplitude for $A(\eta \rightarrow \pi^+ \pi^- \pi^0)$ is known to predict [1] a decay rate well below measurement [7].

The next order contribution to $A^{(IB)}$, $O(p^4)$, involves one-loop diagrams with vertices from (7) and tree-level counter-terms from the non-anomalous part $\sum L_i L_4^{(i)}$ of (10). They lead to the $U+V+W$ terms in eq (13) and their values will be taken from the detailed analysis of ref. [1] introducing two simplifying approximations. On the one hand, we will assume that the smallness of the available phase-space in both $\eta \rightarrow \pi^+ \pi^- \pi^0$ and $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$ decays allows to approximate these corrections with the value at the center of the Dalitz plot for $\eta \rightarrow \pi^+ \pi^- \pi^0$. With $\mu = 0.75 \text{ GeV} \simeq M_\rho \simeq M_\omega$, this amounts to fix $U + V = 0.39 - 0.03$, coming from pion-loop effects, as discussed in detail in [1]. On the other hand, the dominant contribution from the finite part of the counter-terms L_i is known to come [1] from $\eta - \eta'$ mixing effects. In our nonet symmetry context with the conventional mixing angle, eq (6), this amounts to take $1 + W \simeq +\sqrt{2}$. Notice that this value corresponds to a value for W (where $\eta - \eta'$ mixing effects manifest) somewhat larger than that proposed in [1], but that our simplified treatment is essentially free from the criticisms recently rised by Leutwyler in ref. [18]. With these numerical values and eqs (8,9) we obtain $\Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0) = 270 \text{ eV}$, in good agreement with experiment, $\Gamma_{EXP}(\eta \rightarrow \pi^+ \pi^- \pi^0) = 283 \pm 27 \text{ eV}$ [7]. In spite of this agreement, we obviously do not claim that our simplified amplitude improves the original and more detailed ones in [1] and [18]. We have simply achieved a successful parameterization for $\eta \rightarrow \pi^+ \pi^- \pi^0$ from which we expect a reasonable prediction for $A^{(IB)}$ once inserted in eq (13).

A similar treatment of the $\eta \rightarrow \pi^+\pi^-\pi^0$ amplitude can be found in a recent analysis by Baur et al. [19].

For two different cuts in the photon energy, E_γ , we then obtain

$$\begin{aligned}\Gamma^{(IB)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) &= 0.050 \text{ eV} \quad , \quad E_\gamma^{min} = 50 \text{ MeV}, \\ \Gamma^{(IB)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) &= 0.76 \text{ eV} \quad , \quad E_\gamma^{min} = 10 \text{ MeV},\end{aligned}\tag{14}$$

and the bremsstrahlung spectrum shown in Fig 2. Uncertainties affecting these predictions come mainly from the estimate in eq. (9) and from neglected higher order corrections in ChPT, rather than from our simplified treatment of $A(\eta \rightarrow \pi^+\pi^-\pi^0)$. Globally, they should be expected to reach some 20 – 25 %. Our results (14) are consistent with a previous analysis [20] once the two notations are unified.

The Structure Dependent amplitude $A^{(SD)}$

In contrast with the just discussed bremsstrahlung amplitude (13), which is proportional to the isospin violating factor $m_d - m_u$, ChPT predicts the existence of further, isospin conserving contributions to the global $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ amplitude. We now proceed to compute the dominant parts of these contributions and collect them into a structure dependent (or direct emission) amplitude $A^{(SD)}$. We will follow the ChPT analysis of $\eta \rightarrow \pi^0\gamma\gamma$ [9], whose amplitude $A(\eta \rightarrow \pi^0\gamma\gamma)$ – apart from the fact that it receives no inner bremsstrahlung contribution – is closely related to $A^{(SD)}$. Indeed, in $\eta \rightarrow \pi^0\gamma\gamma$ one photon is isoscalar ($G = -$) and the other one is isovector ($G = +$). The former is the analog of the (isoscalar) photon in $A^{(SD)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)$, while the latter – having the ρ^0 quantum numbers – plays the role of the $\pi^+\pi^-$ pair. Isospin symmetry allows to work in the good isospin limit, $m_d = m_u$, and to decompose $A^{(SD)}$ in three terms by cyclically rotating the pion charge indices

$$A^{(SD)}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma) \equiv A^{(SD)} = A^{(SD)}(+, 0, -) + A^{(SD)}(0, -, +) + A^{(SD)}(-, +, 0).\tag{15}$$

The lowest order contribution to $A^{(SD)}$ appears at order four and proceeds through the 6 one-loop diagrams of Fig 3, containing two vertices from L_2 , eq (7). Part of this contribution is proportional to $m_d - m_u$ and was already included in $A^{(IB)}$, but there is a second part – involving exclusively isospin conserving kaon loops – which belongs to $A^{(SD)}$. This part has to be finite since no suitable counterterms are available in (10) for $m_d = m_u$. One finds indeed the finite result

$$A_{(4)}^{(SD)}(+, 0, -) = \frac{e}{3\sqrt{6}\pi^2 f^4} (6Qq + 3Q^2 - 4m_K^2) [(\epsilon p_+)(qp_-) - (\epsilon p_-)(qp_+)] I(m_K^2, Q^2, Qq)\tag{16}$$

with $Q = p_+ + p_-$ and

$$I(m_K^2, Q^2, Qq) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{xy}{m_K^2 - 2Qqxy - Q^2y(1-y)} \quad (17)$$

By itself, this order four contribution leads to

$$\Gamma_{(4)}^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = 0.88 \times 10^{-7} \text{ eV}, \quad (18)$$

well below the old estimates (1) and (2), and suggesting that this is not the dominant contribution to $A^{(SD)}$. The same situation was found when analyzing $O(p^4)$ kaon loops for $\eta \rightarrow \pi^0 \gamma \gamma$ in ChPT [9]. Similarly, the smallness of this lowest order contribution seems to be a reminiscence of the vanishing of the old CA estimate of ref. [6].

At next order, $O(p^6)$, one has contributions coming from tree level (or counterterms), from one loop and from two loops. The former belong to L_6 and their finite part will be obtained from saturation with vector mesons. For further reference, we compute the full VMD amplitude for $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$, which proceeds through the two diagrams shown in Fig 4 (apart from rotations of pion indexes). Ignoring negligible effects from $\Gamma_{\rho, \omega}$ finite widths, one obtains

$$\begin{aligned} A_{VMD}^{(SD)}(+, 0, -) &= \frac{\sqrt{6}eg^4}{4\pi^4 f^2 (M_\rho^2 - Q^2)} \left[P_\rho P_\omega [(\epsilon p_+) ((P p_-)(q p_0) - (P p_0)(q p_-))] + \right. \\ &\quad \left. P_\rho P'_\rho [(\epsilon p_+) ((P p_-)(q p_0) - (P p_0)(q p_-)) - \frac{1}{3}(q p_0) ((\epsilon p_+)(q p_-) - (\epsilon p_-)(q p_+))] \right] \end{aligned} \quad (19)$$

with

$$\begin{aligned} P_\rho P_\omega &= \frac{1}{M_\omega^2 - (P - q)^2} \left(\frac{1}{M_\rho^2 - Q^2} + \frac{1}{M_\rho^2 - Q_+^2} + \frac{1}{M_\rho^2 - Q_-^2} \right) \\ P_\rho P'_\rho &= \frac{1}{M_\rho^2 - Q^2} \frac{1}{M_\rho^2 - (P - Q)^2} + \frac{1}{M_\rho^2 - Q_+^2} \frac{1}{M_\rho^2 - (P - Q_+)^2} + \frac{1}{M_\rho^2 - Q_-^2} \frac{1}{M_\rho^2 - (P - Q_-)^2} \end{aligned}$$

and $Q_\pm = p_\pm + p_0$. The coupling constants are such that $M_\rho^2 \simeq M_\omega^2 \simeq 2f^2 g^2$ and $g = 4.2$, as discussed in refs.[9, 12, 13]. The part of this VMD contribution which corresponds to the L_6 counter-terms in ChPT can be simply obtained from (19) by expanding the VM propagators $1/(M_V^2 - K^2) = 1/M_V^2 + K^2/M_V^4 \dots$ and retaining only the first term. This leads to $\Gamma_{VMD(6)}^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = 0.42 \times 10^{-4} \text{ eV}$, which turns out to be much larger than the lower order estimate (18). Since in ChPT loop corrections at order six are expected to be only a fraction of the corresponding loop corrections at order four, eq (18), we can safely conclude that the full ChPT prediction at order six is drastically dominated by the VM saturated counter-terms, i.e.,

$$\Gamma_{(6)}^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) \simeq \Gamma_{VMD(6)}^{SD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = 0.42 \times 10^{-4} \text{ eV} \quad (20)$$

The same dominance of $O(p^6)$ counterterms over the corresponding loops was also observed in the case of $A(\eta \rightarrow \pi^0 \gamma \gamma)$ [9].

At order $O(p^8)$ more counterterms appear and a new type of loop-correction becomes potentially important. The contributions of resonance dominated counterterms can simply be obtained by expanding the full VMD amplitude (19) and retaining only $O(p^8)$ terms. By itself, this leads to a non negligible order p^8 correction, namely, $\Gamma_{VMD(8)}^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = 0.094 \times 10^{-4} \text{ eV}$ representing a non negligible increase to Eq.(20). The new type of loop effects looks *a priori* more interesting. Taking two vertices from the anomalous L_{WZ} (one from (11) and another from (12)) one obtains a non-anomalous one-loop correction of order p^8 , which does not vanish only for kaon loops. This "doubly-anomalous" loop contributions leads to a decay rate of $4.2 \times 10^{-7} \text{ eV}$, i.e., of the same order as $\Gamma_{(4)}^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma)$ and well below the order eight counterterm. Again, we find that kaon loops for $A^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma)$ closely follows the same pattern as kaon loops for $A(\eta \rightarrow \pi^0 \gamma \gamma)$. The whole order eight contribution alone is therefore also dominated by counterterms,

$$\Gamma_{(8)}^{(SD)}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) \simeq \Gamma_{VMD(8)}^{SD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = 0.094 \times 10^{-4} \text{ eV} \quad (21)$$

All this implies that in a ChPT context with resonance saturated counterterms, the whole structure dependent amplitude for $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$, $A^{(SD)}$, is strongly dominated by the contributions of these counterterms and should essentially be given by the full VMD amplitude (19). Such an "all-order" estimate leads to

$$\Gamma^{SD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) \simeq \Gamma_{VMD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) = 1.4 \times 10^{-4} \text{ eV} \quad (22)$$

The corresponding photonic spectrum is shown in Fig. 2.

This structure dependent part of the amplitude is the one that should be compared to older estimates, although none of those treatments coincides precisely with ours. From eq (22) and the result $\Gamma_{VMD}(\eta \rightarrow \pi^0 \gamma \gamma) = 0.31 \text{ eV}$ of ref. [9], one obtains $\Gamma_{VMD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma) / \Gamma_{VMD}(\eta \rightarrow \pi^0 \gamma \gamma) \simeq 0.44 \times 10^{-3}$ somewhat below the old estimates (1) and above the vanishing prediction of ref. [6]. The shape of our photonic spectrum coincides with the old prediction by Singer [3] who used a simplified version of VMD not far from ours.

Our final ChPT predictions for the *whole* $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$ amplitude can be obtained from the sum of $A^{(IB)}$, eq (13), and the full VMD amplitude, eq (19). The contribution of the latter – in modulus plus interference with $A^{(IB)}$ – has negligible effects for soft photons and represents a minor increase (somewhat below 1 %) of the decay rate in the higher half of the photonic spectrum as shown in Fig. 2. The integrated width remains thus unaffected as in eq (14).

Conclusions

From the preceding and somewhat intricate analysis of the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay, a definite two-fold conclusion emerges, namely, that the whole amplitude is strongly dominated by inner bremsstrahlung and that it is expected to be large enough to allow for detection in a near future. This is a useful conclusion in that it contradicts (and presumably corrects) older results (see refs. [4],[3]) and it will furnish a clear-cut test for (and presumably confirm) ChPT. But it is also a deceptive conclusion reducing most of the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ dynamics to that of $\eta \rightarrow \pi^+\pi^-\pi^0$, already studied both theoretically [1] and experimentally [7] (although not free of uncertainties [1, 18, 21]). Only an extremely sensitive and dedicated experiment, and an improved theoretical treatment of $A^{(IB)}$ could allow to extract the structure dependent part of the amplitude which contains the genuinely new effects of ChPT for $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$. In this case, the values of counterterms and the hypothesis of their resonance saturation – rather than chiral loop effects – will be tested. Stated otherwise, the experimental analysis of $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ can represent an excellent confirmation of ChPT if the predicted spectrum (largely dominated by bremsstrahlung) is observed, but it can hardly be useful to improve our knowledge on other aspects of this theory .

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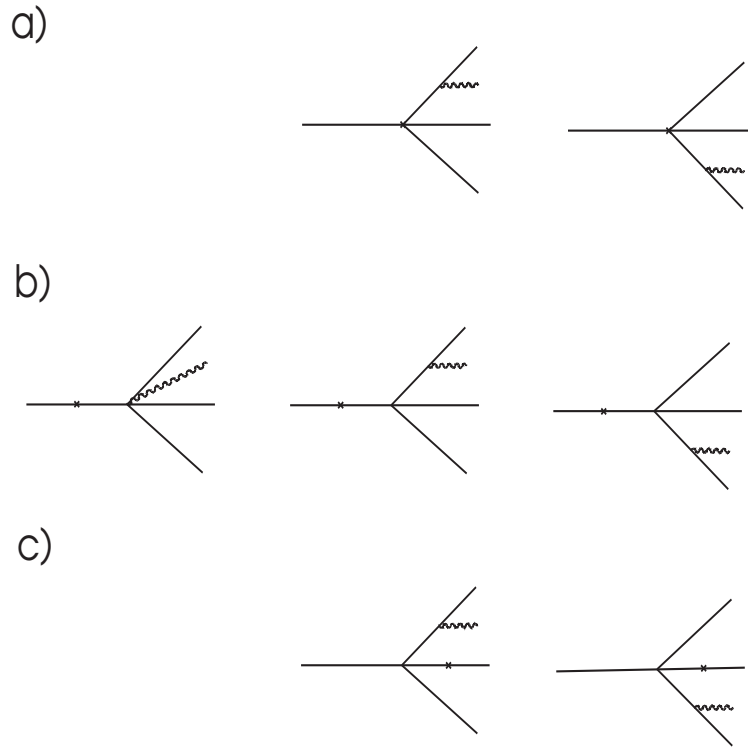


Figure 1: Diagrams contributing to $A^{IB}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)$ at lowest order, $O(p^2)$, in ChPT. The cross in each diagram indicates a coupling proportional to $m_d - m_u$.

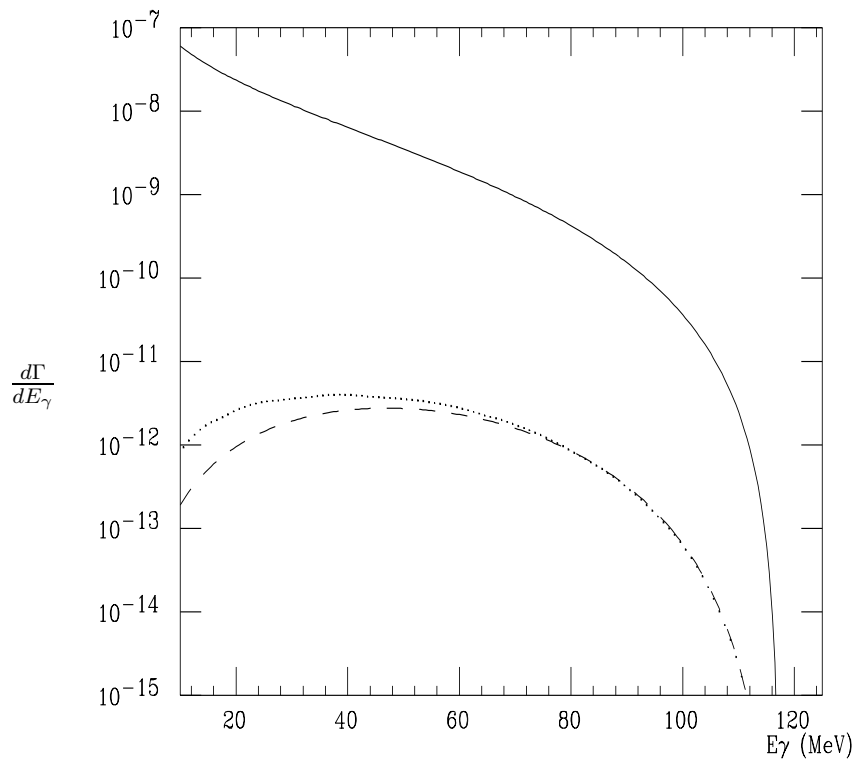


Figure 2: Spectrum of the photon energies, E_γ , as predicted by ChPT for the $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$ decay. The solid line corresponds to the inner bremsstrahlung contribution. The dashed line is the VMD contribution alone. The dotted line is their interference.

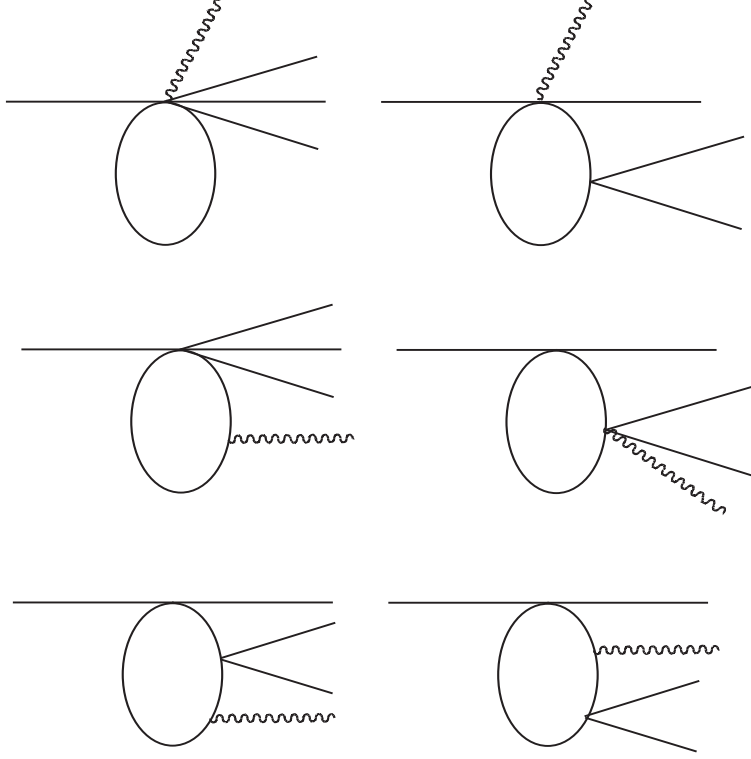


Figure 3: Kaon loops contributing to $A^{SD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma)$ at $O(p^4)$.

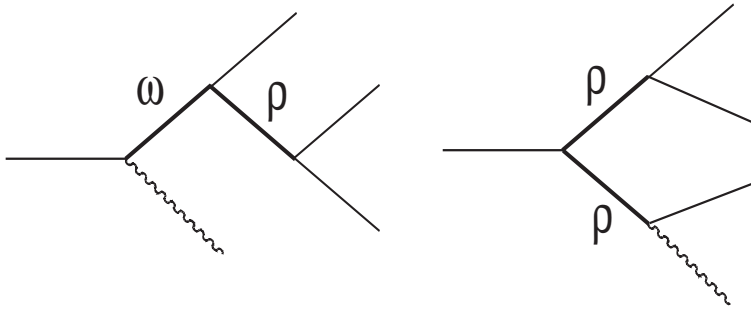


Figure 4: VMD contributions (apart from pion index rotations) to $A^{SD}(\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma)$.